VANISHING CYCLE SHEAVES OF ONE-PARAMETER SMOOTHINGS AND QUASI-SEMISTABLE DEGENERATIONS

ALEXANDRU DIMCA AND MORIHIKO SAITO

Abstract. We study the vanishing cycles of a one-parameter smoothing of a complex analytic space and show that the weight filtration on its perverse cohomology sheaf of the highest degree is quite close to the monodromy filtration so that its graded pieces have a modified Lefschetz decomposition. We describe its primitive part using the weight filtration on the perverse cohomology sheaves of the constant sheaves. As a corollary we show in the local complete intersection case that 1 is not an eigenvalue of the monodromy on the reduced Milnor cohomology at any points if and only if the total space and the singular fiber are both rational homology manifolds. Also we introduce quasi-semistable degenerations and calculate the limit mixed Hodge structure by constructing the weight spectral sequence. As a corollary we show non-triviality of the space of vanishing cycles of the Lefschetz pencil associated with a tensor product of any two very ample line bundles except for the case of even-dimensional projective space where two has to be replaced by three.

Introduction

Let X be a complex analytic space of dimension n, and X_i be the irreducible components of dimension n. Let $\mathrm{IC}_X\mathbf{Q} := \bigoplus_i \mathrm{IC}_{X_i}\mathbf{Q}$ be the intersection complex, which belongs to the category of perverse sheaves $\mathrm{Perv}(X,\mathbf{Q})$ (see [3]) and is naturally identified with a mixed Hodge module ([23], 3.21). Shrinking X if necessary, we assume that X is an intersection of hypersurfaces in a complex manifold so that $\mathbf{Q}_X$ exists in the derived category of mixed Hodge modules, see (1.3) below. By the same argument as in [23], 4.5.9, there is a canonical morphism

$$\mathbf{Q}_X[n] \to \mathrm{IC}_X\mathbf{Q},$$

inducing an isomorphism

$$\mathrm{Gr}^W_n {}^p\mathcal{H}^n\mathbf{Q}_X \xrightarrow{\sim} \mathrm{IC}_X\mathbf{Q}, \tag{0.1}$$

where ${}^p\mathcal{H}^j$ is the perverse cohomology [3] and W is the weight filtration of ${}^p\mathcal{H}^j\mathbf{Q}_X$ defined in the abelian category of mixed Hodge modules $\mathrm{MHM}(X)$, see [23]. (In this paper mixed Hodge modules are denoted by their underlying perverse sheaves when there is no fear of confusions.) It is well-known (see [3], 4.2.4 and [23], 2.26) that

$$ {}^p\mathcal{H}^j\mathbf{Q}_X = 0 \ (j > n), \quad \mathrm{Gr}^W_k {}^p\mathcal{H}^j\mathbf{Q}_X = 0 \ (k > j). \tag{0.2}$$

Date: Aug. 8, 2010, v.3b.

This work is partially supported by Kakenhi 19540023 and by ANR-08-BLAN-0317-02 (SEDIGA). Additionally, A. Dimca is grateful to ASSMS, Government College University, Lahore, Pakistan, where part of the work on this paper was done.

It is also well-known to the specialists that $\mathbf{Q}_X[n] = \mathrm{IC}_X\mathbf{Q}$ if and only if X is a $\mathbf{Q}$-homology manifold, see (1.5) below. Note that the ${}^p\mathcal{H}^j\mathbf{Q}_X$ are globally well-defined in $\mathrm{MHM}(X)$ (see [23], 2.19) and we have by (0.1–2)

$$(0.3) \qquad \mathbf{Q}_X[n] = \mathrm{IC}_X\mathbf{Q} \iff \big[{}^p\mathcal{H}^j\mathbf{Q}_X = 0 \ (j < n),\ \mathrm{Gr}^W_j {}^p\mathcal{H}^n\mathbf{Q}_X = 0 \ (j < n)\big].$$

Let $f : X \to \mathbf{C}$ be a holomorphic function on X such that the fibers $f^{-1}(t)$ are smooth and purely $(n-1)$-dimensional for $t \ne 0$. Setting $Y = f^{-1}(0)$, this condition is essentially equivalent to the smoothness and the pure dimensionality of $X \setminus Y$ replacing X with an open neighborhood of Y if necessary. Assume f is nonconstant on any n-dimensional irreducible component of X. Let $i : Y \hookrightarrow X$ denote the inclusion.

Theorem 1. *Let f be as above. Then we have an exact sequence of mixed Hodge modules on Y*

$$0 \to {}^p\mathcal{H}^{n-1}\mathbf{Q}_X \to {}^p\mathcal{H}^{n-1}\mathbf{Q}_Y \to {}^p\mathcal{H}^{-1}i^*(\mathrm{IC}_X\mathbf{Q}) \to W_{n-1}{}^p\mathcal{H}^n\mathbf{Q}_X \to 0,$$

and isomorphisms

$${}^p\mathcal{H}^j\mathbf{Q}_X = {}^p\mathcal{H}^j\mathbf{Q}_Y \quad (j < n-1).$$

This is shown by applying the functor i^* to the distinguished triangle associated to the mapping cone of $\mathbf{Q}_X \to (\mathrm{IC}_X\mathbf{Q})[-n]$ and using (0.1–2). If $\mathbf{Q}_X[n]$ is a perverse sheaf, then $\mathbf{Q}_Y[n-1]$ is also a perverse sheaf by the last isomorphisms of Theorem 1, and we get

Corollary 1. *If $\mathbf{Q}_X[n]$ is a perverse sheaf, then we have $\mathrm{IC}_X\mathbf{Q} = \mathbf{Q}_X[n]$ if and only if $i^*(\mathrm{IC}_X\mathbf{Q})[-1] = \mathbf{Q}_Y[n-1]$.*

Let ψ_f, φ_f denote the nearby and vanishing cycle functors [6]. We denote by $\psi_{f,1}, \varphi_{f,1}$ their unipotent monodromy part. In this paper these functors are *shifted by* -1 so that they preserve perverse sheaves and commute with ${}^p\mathcal{H}^j$. Set $N = \log T_u$ with T_u the unipotent part of the monodromy T. Note that $\psi_f\mathbf{Q}_X[n] = \psi_f\mathrm{IC}_X\mathbf{Q}$ is a perverse sheaf, and $\psi_f{}^p\mathcal{H}^j\mathbf{Q}_X = 0$ for $j \ne n$, since $X \setminus Y$ is smooth and $\psi_f\mathcal{F}$ depends only on $\mathcal{F}|_{X\setminus Y}$ in general. The weight filtration W on the nearby cycles $\psi_f\mathbf{Q}_X[n] = \psi_f\mathrm{IC}_X\mathbf{Q}$ is the monodromy filtration [7] shifted by $n-1$. However, W on the vanishing cycles $\varphi_{f,1}{}^p\mathcal{H}^j\mathbf{Q}_X$ is rather complicated in case X is singular (even in the isolated singularity case, see e.g. [27]). Since ${}^p\mathcal{H}^j\mathbf{Q}_X$ for $j \ne n$ is supported in Y, $\varphi_{f,1}{}^p\mathcal{H}^j\mathbf{Q}_X$ is identified with ${}^p\mathcal{H}^j\mathbf{Q}_X$ and the action of N on it vanishes. So we are mainly interested in $\varphi_{f,1}{}^p\mathcal{H}^0(\mathbf{Q}_X[n])$. Set

$$V_\bullet = \textstyle\bigoplus_k V_k \quad \text{with} \quad V_k := \mathrm{Gr}^W_k \varphi_{f,1}{}^p\mathcal{H}^0(\mathbf{Q}_X[n]) \in \mathrm{MHM}(Y).$$

Theorem 2. *Let f be as in Theorem* 1. *There is a noncanonical decomposition*

$$V_\bullet \cong V'_\bullet \oplus V''_\bullet \quad \textit{in } \mathrm{MHM}(Y) \textit{ compatible with } N \textit{ and satisfying}$$

$$N^i : V'_{n+i} \xrightarrow{\sim} V'_{n-i}(-i), \quad N^i : V''_{n-1+i} \xrightarrow{\sim} V''_{n-1-i}(-i) \quad \textit{for } i > 0,$$

where $(-i)$ *is the Tate twist* [5]. *Moreover, setting* $K'_k = \operatorname{Ker} N \cap V'_k$ *and* $K''_k = \operatorname{Ker} N \cap V''_k$, *we have*

$$K'_k = (\mathrm{Gr}^W_{k-2}({}^p\mathcal{H}^{n-1}\mathbf{Q}_Y/{}^p\mathcal{H}^{n-1}\mathbf{Q}_X))(-1) \quad (k \le n),$$
$$K''_k = \mathrm{Gr}^W_k {}^p\mathcal{H}^n\mathbf{Q}_X \quad (k \le n-1).$$

Corollary 2. *With the above notation, we have the Lefschetz decompositions*

$$V'_\bullet = \bigoplus_{k\ge 0}\bigoplus_{i=0}^k N^i P'_{n+k}(i), \quad V''_\bullet = \bigoplus_{k\ge 0}\bigoplus_{i=0}^k N^i P''_{n-1+k}(i),$$
$$\text{with} \quad K'_{n-k} = N^k P'_{n+k}(k), \quad K''_{n-1-k} = N^k P''_{n-1+k}(k),$$

where $P'_{n+k} := \operatorname{Ker} N^{k+1} \subset V'_{n+k}$ *denotes the* N*-primitive part, and similarly for* P''_{n-1+k}.

If $\mathbf{Q}_X[n]$ is a perverse sheaf (e.g. if X is a local complete intersection), then ${}^p\mathcal{H}^{n-1}\mathbf{Q}_X$ vanishes in the above formula and $\mathbf{Q}_Y[n-1]$ is also a perverse sheaf by Theorem 1. In the isolated singularity case, Theorem 2 was essentially stated in [21], see also [27].

From Theorem 2 together with (0.3) for X, Y, we deduce

Theorem 3. *Let* f *be as in Theorem* 1. *Assume* $\mathbf{Q}_X[n]$ *is a perverse sheaf* (*e.g.* X *is a local complete intersection*) *so that* $\mathbf{Q}_Y[n-1]$ *is also a perverse sheaf. Let* F_x *denote the Milnor fiber of* f *around* $x \in Y$. *Then*

(a) *The following three conditions are equivalent to each other.*
 (i) $V'_\bullet = 0$,
 (ii) $\mathbf{Q}_Y[n-1] = \mathrm{IC}_Y\mathbf{Q}$, *i.e.* Y *is a* $\mathbf{Q}$*-homology manifold,*
 (iii) W *on* $\varphi_{f,1}(\mathbf{Q}_X[n])$ *is the monodromy filtration shifted by* $n-1$.

(b) *The following three conditions are equivalent to each other.*
 (i) $V''_\bullet = 0$,
 (ii) $\mathbf{Q}_X[n] = \mathrm{IC}_X\mathbf{Q}$, *i.e.* X *is a* $\mathbf{Q}$*-homology manifold,*
 (iii) W *on* $\varphi_{f,1}(\mathbf{Q}_X[n])$ *is the monodromy filtration shifted by* n.

(c) *The following three conditions are equivalent to each other.*
 (i) $\varphi_{f,1}\mathbf{Q}_X = 0$,
 (ii) X *and* Y *are* $\mathbf{Q}$*-homology manifolds,*
 (iii) 1 *is not an eigenvalue of the monodromy on* $\widetilde{H}^j(F_x, \mathbf{Q})$ *for any* j, x.

The assertion (c) says that $\varphi_{f,1}\mathbf{Q}_X$ just contains the information of the difference between $\mathbf{Q}_X[n]$ and $\mathrm{IC}_X\mathbf{Q}$ together with the difference between $\mathbf{Q}_Y[n-1]$ and $\mathrm{IC}_Y\mathbf{Q}$ in this case, see (0.3). (For assertions using the topological methods, see [18].) We have $V''_\bullet \ne 0$, for example, if $f : X \to \mathbf{C}$ is the base change of $g : Z \to \mathbf{C}$ by an m-fold ramified covering of $\mathbf{C}$ such that Z is smooth and an m-th primitive root of unity is an eigenvalue of the monodromy of g. We have $V'_\bullet \ne 0$, for example, in the case where X is smooth, Y has an isolated singularity, and 1 is an eigenvalue of the Milnor monodromy.

Let $z \in Z := \operatorname{supp} V_\bullet \subset Y$ with the inclusion $i_z : \{z\} \hookrightarrow Z$. Let Z_i be the local irreducible components of (Z, z). Using Theorem 2, we get an assertion on the Milnor cohomology in a special case as follows.

Corollary 3. *Let f be as in Theorem 1. Assume Z is a curve and all the eigenvalues of the monodromy around z of the local system $(V_\bullet|_{Z_i\setminus\{z\}})[-1]$ are different from 1 for any i. Then $H^j i_z^* V_\bullet = 0$ for $j \ne 0$, and the assertion of Theorem 2 holds by replacing $V_\bullet$ with the graded pieces of the unipotent monodromy part of the Milnor cohomology $\mathrm{Gr}_\bullet^W H^{n-1}(F_z,\mathbf{Q})_1$, and applying the functor $H^0 i_z^*$ to the mixed Hodge modules appearing in Theorem 2, where*

$$H^0 i_z^* K_k'' = \mathrm{Gr}_k^W H^{-1} i_z^* \mathrm{IC}_X \mathbf{Q}.$$

In case ${}^p\mathcal{H}^{n-1}\mathbf{Q}_X = 0$, we have moreover

$$H^0 i_z^* K_k'(1) = \mathrm{Gr}_{k-2}^W H^{-1} i_z^*\, C(\mathbf{Q}_Y[n-1] \to \mathrm{IC}_Y\mathbf{Q}).$$

Here the mapping cone $C(\mathbf{Q}_Y[n-1] \to \mathrm{IC}_Y\mathbf{Q})$ may be replaced by $\mathrm{IC}_Y\mathbf{Q}$ if $n \ge 3$.

Corollary 4. *With the assumptions of Corollary 3, the maximal size of the Jordan block of the monodromy on $H^{n-1}(F_z,\mathbf{Q})_1$ is the largest number k such that $H^0 i_z^* K'_{n+1-k} \ne 0$ or $H^0 i_z^* K''_{n-k} \ne 0$ if $k > 0$.*

As for the nearby cycles $\psi_{f,1}(\mathbf{Q}_X[n])$, we have the following

Theorem 4. *Let f be as in Theorem 1. There are noncanonical decompositions for $k < n$*

$$\mathrm{Gr}_k^W \psi_{f,1}(\mathbf{Q}_X[n]) \cap \mathrm{Ker}\, N \cong \mathrm{Gr}_k^W({}^p\mathcal{H}^{n-1}\mathbf{Q}_Y/{}^p\mathcal{H}^{n-1}\mathbf{Q}_X) \oplus \mathrm{Gr}_k^W {}^p\mathcal{H}^n\mathbf{Q}_X.$$

Note that the weight filtration W on $\psi_{f,1}(\mathbf{Q}_X[n])$ is the monodromy filtration shifted by $n-1$, and the left-hand side essentially gives the N-primitive part.

As an application of Theorem 4 we construct the weight spectral sequence associated with a *quasi-semistable* degeneration, see (3.1–2). In a typical case, we have the following.

Theorem 5. *Let $L_k\,(1 \le k \le r)$ be line bundles on a smooth proper complex algebraic variety Y of dimension n, and L_0 be the tensor product of $L_k\,(1 \le k \le r)$ where $n, r \ge 2$. Let Y_k be smooth divisors defined by $g_k \in \Gamma(Y, L_k)\,(0 \le k \le r)$. Assume $\bigcup_{k=0}^r Y_k$ is a divisor with normal crossings on Y. Let $X = \{g_1 \cdots g_r = tg_0\} \subset Y \times \mathbf{C}$ with $f : X \to \mathbf{C}$ induced by the second projection where t is the coordinate of $\mathbf{C}$. Then we have a weight spectral sequence degenerating at E_2*

$$E_1^{-k,j+k} = H^j(X_0, \mathrm{Gr}_{n-1+k}^W \psi_{f,1}\mathrm{IC}_X\mathbf{Q}) \Rightarrow H^{j+n-1}(X_\infty, \mathbf{Q}),$$

where $H^\bullet(X_\infty,\mathbf{Q})$ denotes the limit mixed Hodge structure [25], [26]. Moreover, $E_1^{-i,j+i}$ is the direct sum of

$$\Big(\bigoplus_{|I|=i+2l+1} H^{j+n-|I|}(Y_I)(-i-l)\Big) \oplus \Big(\bigoplus_{|I|=i+2l+2} H^{j+n-|I|-1}(Y_I')(-i-l-1)\Big),$$

over $l \ge \max(-i,0)$, where $Y_I' = Y_0 \cap \bigcap_{i\in I} Y_i$ and $Y_I = \bigcap_{i\in I} Y_i$ for $I \subset \{1,\dots,r\}$.

The first part of the E_1-term is the same as the weight spectral sequence in case of a semistable degeneration [26] where X is smooth and X_0 is a reduced divisor with simple normal crossings. The second part of the E_1-term is closely related to the singularities of X. Note that X is *singular* and f is *not* semistable if $n \ge 3$.

Nevertheless we get the weight spectral sequence as above *without* using a blow-up of X to get a (non-reduced) semistable model as in [11], [13].

In case $Y = \mathbf{P}^n$ we can explicitly describe the E_2-term of the weight spectral sequence, i.e. the graded pieces of the weight filtration of the limit mixed Hodge structure, by using the N-primitive decomposition as below.

Corollary 5. *With the notation and the assumptions of Theorem 5, assume $Y = \mathbf{P}^n$. Let $H^{n-1}_{\mathrm{prim}}(X_\infty, \mathbf{Q})$, $H^{n-|I|}_{\mathrm{prim}}(Y_I, \mathbf{Q})$, $H^{n-1-|I|}_{\mathrm{prim}}(Y'_I, \mathbf{Q})$ denote the middle primitive cohomology, and $P_N\mathrm{Gr}^W_{n-1+k}H^{n-1}_{\mathrm{prim}}(X_\infty, \mathbf{Q})$ denote the N-primitive part defined by* $\mathrm{Ker}\, N^{k+1}$ $(k \geq 0)$*. Set $m = \binom{r-1}{n}$.*

Then $P_N\mathrm{Gr}^W_{n-1+k}H^{n-1}_{\mathrm{prim}}(X_\infty, \mathbf{Q})$ for $k \in [0, n-1]$ is given by

$$\begin{array}{ll} \big(\bigoplus_{|I|=k+1} \widetilde{H}^{n-|I|}_{\mathrm{prim}}(Y_I)(-k)\big) \oplus \big(\bigoplus_{|I|=k+2} \widetilde{H}^{n-1-|I|}_{\mathrm{prim}}(Y'_I)(-k-1)\big) & \text{if } k \neq n-1, \\ \big(\bigoplus_{|I|=k+1} \widetilde{H}^{n-|I|}_{\mathrm{prim}}(Y_I)(-k)\big) \oplus \big(\bigoplus^m \mathbf{Q}(1-n)\big) & \text{if } k = n-1. \end{array}$$

Here $\widetilde{H}^{n-|I|}_{\mathrm{prim}}(Y_I) = 0$ if $|I| > \min(n, r)$, and $\widetilde{H}^{n-1-|I|}_{\mathrm{prim}}(Y'_I) = 0$ if $|I| > \min(n-1, r)$. Let d_k be the integers such that $L_k = \mathcal{O}_{\mathbf{P}^n}(d_k)\,(0 \leq k \leq r)$. Note that $d_0 = \sum_{k=1}^r d_k$. If $d_k = 1$ for any $r \geq 1$ as in [11], then $\widetilde{H}^{n-|I|}_{\mathrm{prim}}(Y_I) = 0$ for any I, and $\mathrm{Gr}^W_{n-1+k}H^{n-1}(X_\infty, \mathbf{Q})$ has level $< n-1-|k|$ for $|k| \neq n-1$ (where the level means the difference between the maximum and the minimum of the p with $\mathrm{Gr}^p_F \neq 0$). Corollary 5 also implies that $N^{n-1} \neq 0$ on $H^{n-1}(X_\infty, \mathbf{Q})$ for $r \geq n+1$, since

$$N^k : \mathrm{Gr}^W_{n-1+k}H^{n-1}(X_\infty, \mathbf{Q}) \xrightarrow{\sim} \mathrm{Gr}^W_{n-1-k}H^{n-1}(X_\infty, \mathbf{Q})(-k) \quad (k > 0).$$

For $j \neq n-1$, we have $H^j(X_\infty, \mathbf{Q}) \cong \mathbf{Q}(-j/2)$ or 0 in the case of Corollary 5. It is easy to show that the direct sums in Corollary 5 are direct factors of $P_N\mathrm{Gr}^W_{n-1+k}H^{n-1}_{\mathrm{prim}}(X_\infty, \mathbf{Q})$. Using the N-primitive decomposition, Corollary 5 is then equivalent to the numerical equality

$$P_n(d_0) = \textstyle\sum_{|I|\geq 1} |I|\binom{r}{|I|} P_n(\mathbf{d}_I) + \sum_{|I|\geq 2}(|I|-1)\binom{r}{|I|} P_n(\mathbf{d}'_I) + n\binom{r-1}{n}, \tag{0.4}$$

where $\mathbf{d}_I = (d_k)_{k\in I}$, $\mathbf{d}'_I = (d_k)_{k\in I\cup\{0\}}$, and $P_n(\mathbf{d})$ denotes the dimension of the reduced middle primitive cohomology of a smooth complete intersection of multidegree $\mathbf{d} \in \mathbf{Z}^k_{>0}$ in $\mathbf{P}^n$. It is possible to show directly (0.4) in some simple cases, see Remark (3.5)(i). By Corollary 5, we can calculate the limit mixed Hodge structure without using a blow-up of X in this case. It generalizes a calculation of the limit mixed Hodge structure for $n = 4$, $d_0 = r = 5$ in [11] where a blow-up of X is used. However, we cannot calculate the group of connected components of a fiber of the Néron model as in loc. cit. by using our method.

As an application of Theorem 5 where $r = 2$, we get the following

Theorem 6. *Let Y be a smooth complex projective variety of dimension $n \geq 2$. Let L be a very ample line bundle defining a closed embedding $Y \hookrightarrow \mathbf{P}^N$. Assume L is a tensor product of k very ample line bundles where $k = 3$ if Y is projective space of even dimension, and $k = 2$ otherwise. Then the vanishing cycles of a Lefschetz pencil do not vanish, i.e. the restriction morphism $i_s^* : H^{n-1}(Y, \mathbf{Q}) \to H^{n-1}(Y_s, \mathbf{Q})$*

is non-surjective where $i_s : Y_s \to Y$ *is a general member of the linear system* $|L| = (\mathbf{P}^N)^\vee$.

This is an improvement of [15], Cor. 6.4 where the assertion was shown for $L = L'^{\otimes d}$ with L' ample and $d \gg 1$. It can be used to show the vanishing of some direct factor of the decomposition in [4] for the direct image of the constant sheaf by $\mathcal{Y} \to (\mathbf{P}^N)^\vee$ where $\mathcal{Y}$ is the total space of the universal family of the hyperplane sections of $Y \subset \mathbf{P}^N$. Note that non-surjectivity of i_s^* implies that the discriminant of π (i.e. the dual variety of Y in $(\mathbf{P}^N)^\vee$) has codimension 1. The converse is true in the n odd case, using the fact that the eigenvalue of the local monodromy of the vanishing cycle is -1 by the Picard-Lefschetz formula [17], see [15], Th. 6.3. Note also that taking the tensor product of two very ample line bundles corresponds to the composition with the Segre embedding. It is possible to prove Theorem 6 by using some arguments in [13], [20].

Recently we are informed that our paper is closely related to some results in [1] and [2].

In Section 1, we recall some basics of mixed Hodge modules and show a lemma used in the proof of Theorem 2. In Section 2, we prove Theorems 1–4 and Corollaries 3–4. In Section 3, we introduce quasi-semistable degenerations and prove a generalization of Theorem 5 and also Corollary 5. In Section 4, we show Theorem 6.

1. Preliminaries

1.1. Weight filtration. Every mixed Hodge module M on a complex analytic space X has a canonical weight filtration W in the category of mixed Hodge modules $\mathrm{MHM}(X)$, and every morphism of mixed Hodge modules is strictly compatible with the weight filtration W. We say that M is pure of weight n if $\mathrm{Gr}_k^W M = 0$ for $k \neq n$. If a mixed Hodge module M is pure, then it has a strict support decomposition $M = \bigoplus_Z M_Z$, where Z runs over the irreducible closed analytic subspaces of X, and M_Z has strict support Z, i.e. its support is Z and there is no nontrivial sub nor quotient object with strictly smaller support. Pure Hodge modules are semisimple since they are assumed to be polarizable, see [22], 5.1–2.

1.2. Nearby and vanishing cycle functors. With the above notation, assume X is pure dimensional, and let X_i be the irreducible components of X. Let M be a pure Hodge module of weight n which is a direct sum of pure Hodge modules M_{X_i} with strict support X_i. Let f be a holomorphic function on X which is nonconstant on any X_i. By definition ([22], 5.1.6) the weight filtration W on the nearby and vanishing cycles $\psi_f M, \varphi_{f,1} M$ is the monodromy filtration shifted by $n-1$ and n respectively. So we have

$$\begin{aligned} N^k &: \mathrm{Gr}^W_{n-1+k}\psi_f M \xrightarrow{\sim} (\mathrm{Gr}^W_{n-1-k}\psi_f M)(-k), \\ N^k &: \mathrm{Gr}^W_{n+k}\varphi_{f,1} M \xrightarrow{\sim} (\mathrm{Gr}^W_{n-k}\varphi_{f,1} M)(-k). \end{aligned} \tag{1.2.1}$$

As for the non-unipotent monodromy part, we have

$$(\psi_{f,\neq 1} M, W) = (\varphi_{f,\neq 1} M, W).$$

Set $Y = f^{-1}(0)$ with the inclusion $i : Y \hookrightarrow X$. Since M has strict support X and $Y \neq X$, we have

$$ {}^p\mathcal{H}^j i^* M = 0 \quad \text{for} \quad j \neq -1, $$

and there is a short exact sequence of mixed Hodge modules on Y

$$ 0 \to {}^p\mathcal{H}^{-1} i^* M \to \psi_{f,1} M \to \varphi_{f,1} M \to 0. \tag{1.2.2} $$

In fact, the functor i^* is defined by the mapping cone of $\psi_{f,1} \to \varphi_{f,1}$, i.e. we have a distinguished triangle

$$ i^*[-1] \to \psi_{f,1} \to \varphi_{f,1} \to . \tag{1.2.3} $$

This is slightly different from the usual one since $\psi_{f,1}, \varphi_{f,1}$ in this paper are shifted by -1 so that they preserve perverse sheaves. We have moreover isomorphisms

$$ {}^p\mathcal{H}^{-1} i^* M = \operatorname{Ker} N \subset \psi_{f,1} M, \quad \varphi_{f,1} M = \operatorname{Coim} N, \tag{1.2.4} $$

where $N : \psi_{f,1} M \to \psi_{f,1} M(-1)$, see [22], 5.1.4. Indeed, we have a surjection and an injection $\operatorname{can} : \psi_{f,1} M \twoheadrightarrow \varphi_{f,1} M$ and $\operatorname{Var} : \varphi_{f,1} M \hookrightarrow \psi_{f,1} M(-1)$ such that $N = \operatorname{Var} \circ \operatorname{can}$.

1.3. Constant sheaf case. If X is an intersection of hypersurfaces in a complex manifold V (shrinking X if necessary), then we have $\mathbf{Q}_X$ in the derived category of mixed Hodge modules $\mathcal{D} := D^b\mathrm{MHM}(X)$ as in the proof of Prop. 2.19 in [23]. Indeed, if $i_X : X \hookrightarrow V$ and $i_j : g^{-1}(0) \to V$ denote the inclusions where $\bigcap_j g^{-1}(0) = X$, then

$$ (i_X)_* i_X^* = \textstyle\prod_j (i_j)_* i_j^* \quad \text{with} \quad (i_j)_* i_j^* = C(\psi_{g_j,1} \to \varphi_{g_j,1}). $$

Note that (0.1) is equivalent to

$$ \begin{aligned} \operatorname{Hom}_{\mathcal{D}}(\mathrm{Gr}_n^W {}^p\mathcal{H}^n \mathbf{Q}_X, M) = \operatorname{Hom}_{\mathcal{D}}({}^p\mathcal{H}^n \mathbf{Q}_X, M) &= \operatorname{Hom}_{\mathcal{D}}(\mathbf{Q}_X[n], M) \\ &= \operatorname{Ext}_{\mathcal{D}}^{\dim Z - n}(\mathbf{Q}_Z[\dim Z], M) = 0 \end{aligned} $$

for any pure Hodge module M of weight n with $Z := \operatorname{supp} M \subset \operatorname{Sing} X$ (where we may assume that Z is an intersection of hypersurfaces shrinking V if necessary). Indeed, the strict support decomposition implies

$$ \mathrm{Gr}_n^W {}^p\mathcal{H}^n \mathbf{Q}_X = \mathrm{IC}_X \mathbf{Q} \oplus M', $$

where M' is a pure Hodge module of weight n and $\operatorname{supp} M' \subset \operatorname{Sing} X$, and it is enough to show that $M' = 0$.

For a holomorphic function f on X and $x \in Y := f^{-1}(0)$, let $i_x : \{x\} \hookrightarrow Y$ denote the inclusion, and F_x denote the Milnor fiber around x. Since ψ_f and φ_f in this paper are shifted by -1, we have

$$ \begin{aligned} H^j i_x^* \psi_f(\mathbf{Q}_X[n]) &= H^{n-1+j}(F_x, \mathbf{Q}), \\ H^j i_x^* \varphi_f(\mathbf{Q}_X[n]) &= \widetilde{H}^{n-1+j}(F_x, \mathbf{Q}). \end{aligned} \tag{1.3.1} $$

The following will be used in the proof of Theorem 2.

Lemma 1.4. *Let $\mathcal{A}$ be the category consisting of $(M_\bullet, N)$ where $M_\bullet = \bigoplus_{k \in \mathbf{Z}} M_k$ with M_k a pure Hodge module of weight k and $N : M_\bullet \to M_\bullet(-1)$ is a morphism of graded Hodge modules (here $(M_\bullet(-1))_k := M_{k-2}(-1)$.) Morphisms of $\mathcal{A}$ are*

morphisms of graded Hodge modules compatible with the action of N. Assume there is a commutative diagram of exact sequences in $\mathcal{A}$

$$\begin{array}{ccccccccc} & & 0 & & 0 & & C'_\bullet & & \\ & & \downarrow & & \downarrow & & \bigcap & & \\ 0 & \to & A'_\bullet & \to & B_\bullet & \to & C_\bullet & \to & 0 \\ & & \bigcap & & || & & \twoheaddownarrow & & \\ 0 & \to & A_\bullet & \to & B_\bullet & \to & C''_\bullet & \to & 0 \\ & & \twoheaddownarrow & & \downarrow & & \downarrow & & \\ & & A''_\bullet & & 0 & & 0 & & \end{array}$$

such that

$$N^k : B_{n-1+k} \xrightarrow{\sim} B_{n-1-k}(-k) \quad (k > 0),$$
$$A_\bullet = \mathrm{Ker}(N : B_\bullet \to B_\bullet(-1)).$$

Then, choosing a graded splitting of $A_\bullet \twoheadrightarrow A''_\bullet$, we have an isomorphism in $\mathcal{A}$

$$C_\bullet = (\textstyle\bigoplus_{k\ge1}\bigoplus_{j=1}^k A'_{n-1-k}(-j)) \oplus (\bigoplus_{k\ge0}\bigoplus_{j=0}^k A''_{n-1-k}(-j)),$$

such that $N : C_\bullet \to C_\bullet(-1)$ is identified with a morphism induced by the identity on $A'_{n-1-k}(-j)$, $A''_{n-1-k}(-j)$.

Proof. Note first that $C'_\bullet = A''_\bullet$ by the snake lemma. Since $A_\bullet$ is identified with the N-primitive part of $B_\bullet$ up to Tate twists, we have the Lefschetz decompositions

$$B_\bullet = \textstyle\bigoplus_{k\ge0}\bigoplus_{j=0}^k A_{n-1-k}(-j), \quad C''_\bullet = \bigoplus_{k\ge1}\bigoplus_{j=1}^k A_{n-1-k}(-j),$$

such that $N : B_\bullet \to B_\bullet(-1)$ is identified with a morphism induced by the identity on $A_{n-1-k}(-j)$, and similarly for $C''_\bullet$. Then, choosing a graded splitting of $A_\bullet \twoheadrightarrow A''_\bullet$, the assertion follows.

Remark 1.5. It is well-known to the specialists that the condition $\mathbf{Q}_X[n] = \mathrm{IC}_X\mathbf{Q}$ is equivalent to that X is a $\mathbf{Q}$-homology manifold. Indeed, the former condition implies that

$$\mathbf{Q}_X[n] = (\mathbf{D}\mathbf{Q}_X)(-n)[-n], \tag{1.5.1}$$

using the self-duality $\mathbf{D}(\mathrm{IC}_X\mathbf{Q}) = \mathrm{IC}_X\mathbf{Q}(n)$ where $\mathbf{D}$ denotes the functor associating the dual. Then (1.5.1) implies that X is a $\mathbf{Q}$-homology manifold. The converse is easy, see also [10], [18].

2. Proof of Theorems 1–4 and Corollaries 3–4

2.1. Proof of Theorems 1 and 4. Set $\mathcal{G}_X = C(\mathbf{Q}_X \to (\mathrm{IC}_X\mathbf{Q})[-n])[-1]$ so that we have a distinguished triangle

$$\mathcal{G}_X \to \mathbf{Q}_X \to (\mathrm{IC}_X\mathbf{Q})[-n] \to . \tag{2.1.1}$$

Since the ${}^p\mathcal{H}^j\mathcal{G}_X$ are supported on Y, $\mathcal{G}_X$ can be identified with a complex of mixed Hodge modules on Y (i.e. it is viewed as an abbreviation of $i_*\mathcal{G}_X$), see [23], 2.23. Applying the functor i^*, we get then a distinguished triangle on Y

$$\mathcal{G}_X \to \mathbf{Q}_Y \to i^*(\mathrm{IC}_X\mathbf{Q})[-n] \to . \tag{2.1.2}$$

Since f is nonconstant on any n-dimensional irreducible component of X, we have

$$ {}^p\mathcal{H}^j i^*(\mathrm{IC}_X\mathbf{Q}) = 0 \quad (j \ne -1). \tag{2.1.3}$$

By (2.1.1) and (0.1–2) we have

(2.1.4)

$$ {}^p\mathcal{H}^j\mathcal{G}_X \xrightarrow{\sim} {}^p\mathcal{H}^j\mathbf{Q}_X \ (j<n), \quad {}^p\mathcal{H}^n\mathcal{G}_X \xrightarrow{\sim} W_{n-1}{}^p\mathcal{H}^n\mathbf{Q}_X, \quad {}^p\mathcal{H}^j\mathcal{G}_X = 0 \ (j>n),$$

and Theorem 1 follows by using the long exact sequence associated to the distinguished triangle (2.1.2). Then Theorem 4 follows from (1.2.4) and Theorem 1 together with the semisimplicity of pure Hodge modules.

2.2. Proof of Theorem 2. Applying (1.2.3) to (2.1.1) shifted by n, and taking the associated long exact sequences, we get a commutative diagram of exact sequences

$$\begin{array}{ccccccc}
0 & & 0 & & & & 0 \\
\downarrow & & \downarrow & & & & \downarrow \\
{}^p\mathcal{H}^{n-1}\mathcal{G}_X & \xrightarrow{\sim} & {}^p\mathcal{H}^{n-1}\mathcal{G}_X & \to & 0 & \to & {}^p\mathcal{H}^n\mathcal{G}_X \\
\| & & \downarrow & & \downarrow & & \downarrow \\
\varphi_{f,1}{}^p\mathcal{H}^{n-1}\mathbf{Q}_X & \hookrightarrow & {}^p\mathcal{H}^{n-1}\mathbf{Q}_Y & \to & \psi_{f,1}{}^p\mathcal{H}^n\mathbf{Q}_X & \twoheadrightarrow & \varphi_{f,1}{}^p\mathcal{H}^n\mathbf{Q}_X \\
\downarrow & & \downarrow & & \| & & \downarrow \\
0 & \to & {}^p\mathcal{H}^{-1}i^*(\mathrm{IC}_X\mathbf{Q}) & \hookrightarrow & \psi_{f,1}(\mathrm{IC}_X\mathbf{Q}) & \twoheadrightarrow & \varphi_{f,1}(\mathrm{IC}_X\mathbf{Q}) \\
\downarrow & & \twoheaddownarrow & & \downarrow & & \downarrow \\
{}^p\mathcal{H}^n\mathcal{G}_X & \xrightarrow{\sim} & {}^p\mathcal{H}^n\mathcal{G}_X & \to & 0 & & 0
\end{array}$$

where all the squares commute since we have the vanishing of certain terms of the the squares of anti-commutativity (see [3], 1.1.11). Here $\mathcal{G}_X$ is identified with $i_*\mathcal{G}_X$ so that $\varphi_{f,1}\mathcal{G}_X$ and $i^*\mathcal{G}_X$ are identified with $\mathcal{G}_X$. The fourth row for $\mathrm{IC}_X\mathbf{Q}$ is a short exact sequence since f is nonconstant on any irreducible component X_i of X and $\mathrm{IC}_{X_i}\mathbf{Q}$ has strict support X_i, see (1.2.2). By (2.1.4) the above diagram induces a diagram of the snake lemma

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & W_{n-1}{}^p\mathcal{H}^n\mathbf{Q}_X & & \\
 & & \downarrow & & \downarrow & & \bigcap & & \\
0 & \to & {}^p\mathcal{H}^{n-1}\mathbf{Q}_Y/{}^p\mathcal{H}^{n-1}\mathbf{Q}_X & \to & \psi_{f,1}{}^p\mathcal{H}^n\mathbf{Q}_X & \to & \varphi_{f,1}{}^p\mathcal{H}^n\mathbf{Q}_X & \to & 0 \\
 & & \bigcap & & \| & & \twoheaddownarrow & & \\
0 & \to & {}^p\mathcal{H}^{-1}i^*(\mathrm{IC}_X\mathbf{Q}) & \to & \psi_{f,1}(\mathrm{IC}_X\mathbf{Q}) & \to & \varphi_{f,1}(\mathrm{IC}_X\mathbf{Q}) & \to & 0 \\
 & & \twoheaddownarrow & & \downarrow & & \downarrow & & \\
 & & W_{n-1}{}^p\mathcal{H}^n\mathbf{Q}_X & & 0 & & 0 & &
\end{array}$$

The weight filtration W on $\psi_{f,1}(\mathrm{IC}_X\mathbf{Q})$ and $\varphi_{f,1}(\mathrm{IC}_X\mathbf{Q})$ are the monodromy filtrations shifted by $n-1$ and n respectively since $\mathrm{IC}_X\mathbf{Q}$ is pure of weight n, see (1.2.1). Moreover, we have by (1.2.4)

$$ {}^p\mathcal{H}^{-1}i^*(\mathrm{IC}_X\mathbf{Q}) = \mathrm{Ker}\, N \subset \psi_{f,1}(\mathrm{IC}_X\mathbf{Q}).$$

So the assertion follows from Lemma (1.4).

2.3. Proof of Theorem 3. Since $\widetilde{H}^{n-1+j}(F_x,\mathbf{Q})_1 = \mathcal{H}^j(\varphi_{f,1}\mathbf{Q}_X[n])_x$, Theorem 3 follows from Theorem 2 together with (0.3) for X, Y.

2.4. Proof of Corollaries 3 and 4. The graded pieces

$$V_k := \mathrm{Gr}_k^W \varphi_{f,1} {}^p\mathcal{H}^0(\mathbf{Q}_X[n])$$

have strict support decompositions

$$V_k = (\textstyle\bigoplus_i (V_k)_{Z_i}) \oplus (\bigoplus_{z\in Z} (V_k)_{\{z\}}),$$

where Z_i are the irreducible components of Z. By the assumption on the local monodromy $T_{i,z}$ along Z_i around z, we have

$$H^j i_z^*(V_k)_{Z_i} = 0 \quad \text{for any } \ j,$$

since it is calculated by the mapping cone of the action of $T_{i,z} - id$. On the other hand we have clearly

$$H^j i_z^*(V_k)_{\{z\}} = 0 \quad \text{for } \ j \neq 0,$$

These imply the E_1-degeneration of the spectral sequence associated to the functor $H^\bullet i_z^*$ and the filtration W on ${}^p\mathcal{H}^0\varphi_{f,1}(\mathbf{Q}_X[n])$, i.e.

$$H^\bullet i_z^* \text{ and } \mathrm{Gr}_\bullet^W \text{ commute on } {}^p\mathcal{H}^0\varphi_{f,1}(\mathbf{Q}_X[n]).$$

We have moreover

$$H^j i_z^*({}^p\mathcal{H}^0\varphi_{f,1}(\mathbf{Q}_X[n])) = \begin{cases} H^{n-1}(F_z, \mathbf{Q})_1 & \text{if } \ j = 0, \\ 0 & \text{if } \ j \neq 0. \end{cases}$$

Indeed, the assertion for $i = 0$ follows from the spectral sequence

$$\begin{aligned} E_2^{i,j} = H^i i_z^*({}^p\mathcal{H}^j\varphi_{f,1}(\mathbf{Q}_X[n])) \Rightarrow H^{i+j} i_z^*(\varphi_{f,1}(\mathbf{Q}_X[n])) \\ = H^{i+j+n-1}(F_z, \mathbf{Q})_1, \end{aligned}$$

since $E_2^{i,j} = 0$ if $i > 0$ or $i < 0, j = 0$. We have a similar spectral sequence with $\varphi_{f,1}(\mathbf{Q}_X[n])$ replaced by $\mathcal{G}_X$, since $\mathrm{Gr}_k^W {}^p\mathcal{H}^n\mathcal{G}_X$ is a direct factor of V_k'' by Theorem 2. So we get similarly

$$H^j i_z^*({}^p\mathcal{H}^n\mathcal{G}_X) = \begin{cases} H^n i_z^*\mathcal{G}_X & \text{if } \ j = 0, \\ 0 & \text{if } \ j \neq 0. \end{cases}$$

Here we have

$$H^n i_z^*\mathcal{G}_X = H^{-1} i_z^* \mathrm{IC}_X\mathbf{Q},$$

since $n \geq 2$ (because $\dim Z = 1$). In case ${}^p\mathcal{H}^{n-1}\mathbf{Q}_X = 0$, we have a similar argument with X, n replaced by $Y, n-1$, since we have the isomorphism

$$K_k' = (\mathrm{Gr}_{k-2}^W {}^p\mathcal{H}^{n-1}\mathbf{Q}_Y)(-1) \ \ (k \leq n).$$

So Corollaries 3–4 follow from Theorem 2.

Remarks 2.5. (i) In the case X is a $\mathbf{Q}$-homology manifold, Theorem 3 implies that Y is a $\mathbf{Q}$-homology manifold (i.e. $\mathbf{Q}_Y[n-1] = \mathrm{IC}_Y\mathbf{Q}$) if and only if $\varphi_{f,1}\mathbf{Q}_X = 0$. This seems to be known to the specialists at least if X is smooth, see also [28]. Note that this equivalence does not hold for the singular case even if X is a complete intersection so that ${}^p\mathcal{H}^i(\mathbf{Q}_X[n]) = {}^p\mathcal{H}^i(\mathbf{Q}_Y[n-1]) = 0$ for $i < 0$, see Theorem 3.

(ii) If x is an isolated singularity of X, Y, let $L_{X,x}$ denote the link of (X, x), and similarly for Y. In this case $\mathcal{G}_X$ in (2.1.1) is supported on $\{x\}$, and we have isomorphisms for $j \le n$

$$ {}^p\mathcal{H}^j\mathcal{G}_X = \mathcal{H}^j\mathcal{G}_X = H^{j-1}((\mathrm{IC}_X\mathbf{Q}[-n])_x/\mathbf{Q}) = \widetilde{H}^{j-1}(L_{X,x}, \mathbf{Q}) = H^j_{\{x\}}\mathbf{Q}_X. $$

Combining this with (2.1.4), we see that ${}^p\mathcal{H}^j\mathbf{Q}_X$ and ${}^p\mathcal{H}^j\mathbf{Q}_Y$ in Theorem 1 can be replaced with the local cohomology groups in this case, see also [21], [27].

(iii) If X is a complete intersection and x is an isolated singularity of X, Y, then it is also possible to prove Theorem 3(c) as follows. We have an exact sequence with the notation of Remark (ii) above

$$ 0 \to H^{n-2}(L_{Y.x}) \to H_c^{n-1}(L_{X.x} \setminus L_{Y,x}) \to H^{n-1}(L_{X.x}) \xrightarrow{i'^*} H^{n-1}(L_{Y.x}), $$

where $i' : L_{Y.x} \to L_{X,x}$ is the inclusion, and the morphism i'^* vanishes since $H^{n-1}(L_{X.x})$ has weights $\le n-1$ and $H^{n-1}(L_{Y.x})$ has weights $> n-1$, see e.g. [9]. (It does not seem easy to prove this vanishing without using Hodge or ℓ-adic theory.) Then the assertion follows from the Wang sequence associated to the Milnor fibration $L_{X.x} \setminus L_{Y,x} \to S^1$ constructed in [14].

3. Case of quasi-semistable degenerations

In this section we introduce quasi-semistable degenerations, and prove a generalization of Theorems 5 and also Corollary 5.

3.1. Quasi-semistable degenerations. Let $f : X \to S$ be a proper morphism of complex analytic spaces such that S is an open disc. We say that f is a quasi-semistable degeneration if $X_0 := f^{-1}(0)$ is a reduced variety with simple normal crossings, (X, x) for each $x \in X_0 \cap \mathrm{Sing}\, X$ is analytically isomorphic to

$$ (h^{-1}(0), 0) \subset (\mathbf{C}^{n+1}, 0) \quad \text{with} \quad h = y_1 \cdots y_k - y_n t, \tag{3.1.1} $$

and moreover f is locally identified with t by choosing a local coordinate of S. Here $y_1, \dots, y_n, t$ are the coordinates of $\mathbf{C}^{n+1}$, and $k \in [2, n-1]$ may depend on x. Note that we have on a neighborhood of $x \in X_0 \cap \mathrm{Sing}\, X$

$$ \mathrm{Sing}\, X = \mathrm{Sing}\, X_0 \cap \{y_n = 0\} \subset X_0. \tag{3.1.2} $$

Let $Y_k\,(1 \le k \le r)$ be the irreducible components of X_0 which are smooth by hypothesis. We assume X_0 algebraic (or Kähler in a generalized sense that each Y_i has a Kähler form such that their cohomology classes come from a cohomology class on X_0). Set

$$ Y_I = \textstyle\bigcap_{k \in I} Y_k, \quad Y'_I = Y_I \cap \mathrm{Sing}\, X \quad \text{for} \quad I \subset \{1, \dots, r\}. $$

When we consider Y'_I, we will assume $|I| \ge 2$ so that $Y_I \subset \mathrm{Sing}\, X_0$ and

$$ Y'_I = Y_I \cap \{y_n = 0\} \quad \text{locally}. $$

Since X is locally a hypersurface, $\mathbf{Q}_X[n], \mathbf{Q}_{X_0}[n-1]$ are perverse sheaves. By (3.1.2) we have

$$ \psi_{f,1}\mathrm{IC}_X\mathbf{Q} = \psi_{f,1}(\mathbf{Q}_X[n]) = \psi_f(\mathbf{Q}_X[n]), \tag{3.1.3} $$

where the last isomorphism follows from the fact that the monodromy on the Milnor cohomology of f at $x \in X_0$ is the identity. Indeed, with the notation of (3.1.1), we have a geometric monodromy induced by the action of $\mathbf{R}$ defined by

$$\alpha : (y_1, \dots, y_n, t) \mapsto (e^{2\pi i\alpha} y_1, y_2, \dots, y_n, e^{2\pi i\alpha} t),$$

and it is the identity for $\alpha = 1$.

The advantage of quasi-semistable degeneration is that it is quite easy to construct examples as is shown in Theorem 5 and Corollary 5. To get further a (nonreduced) semistable model we would need blowing-ups as in [11], [13].

Theorem 3.2. *The conclusion of Theorem* 5 *holds for any quasi-semistable degeneration.*

Proof. By (3.1.3) the spectral sequence is associated with the weight filtration W on $\psi_{f,1}\mathrm{IC}_X\mathbf{Q}$ which is the monodromy filtration shifted by $n-1$. By [23], 2.14 (together with the definition of ψ_t), we have isomorphisms as mixed Hodge structures

$$H^j(X_0, \psi_f(\mathbf{Q}_X[n])) = \psi_t{}^pR^jf_*(\mathbf{Q}_X[n])) = H^{j+n-1}(X_\infty, \mathbf{C}),$$

where ψ_f, ψ_t are shifted by -1 so that they preserve perverse sheaves and mixed Hodge modules. By Theorem 4, it is then sufficient to show the following isomorphisms for $k < n$

$$\begin{aligned} \mathrm{Gr}_k^W(\mathbf{Q}_{X_0}[n-1]) &= \textstyle\bigoplus_{|I|=n-k} \mathbf{Q}_{Y_I}[\dim Y_I], \\ \mathrm{Gr}_k^W(\mathbf{Q}_X[n]) &= \textstyle\bigoplus_{|I|=n-k+1} \mathbf{Q}_{Y'_I}[\dim Y'_I](-1). \end{aligned} \tag{3.2.1}$$

The first isomorphisms are well-known. For the second isomorphisms, we use the function h in (3.1.1) defining locally X. By (1.2.4) and using the N-primitive decomposition, there are isomorphisms for $k < n$

$$\begin{aligned} \mathrm{Gr}_k^W(\mathbf{Q}_X[n]) &= \mathrm{Gr}_k^W\psi_{h,1}(\mathbf{Q}_{X'}[n+1]) \cap \mathrm{Ker}\, N \\ &= \mathrm{Gr}_{k+2}^W\varphi_{h,1}(\mathbf{Q}_{X'}[n+1])(1) \cap \mathrm{Ker}\, N. \end{aligned}$$

The assertion is then reduced to the first isomorphisms of (3.2.1) by using Lemma (3.3) below. More precisely, it proves the assertion locally in classical topology, and we have to show that the direct factors of $\mathrm{Gr}_k^W(\mathbf{Q}_X[n])$ $(k < n)$ are *globally* constant sheaves. This is inductively reduced to the case $k = n-1$. (Indeed, the local extension class between the $\mathrm{Gr}_k^W(\mathbf{Q}_X[n])$ induces the restriction morphisms $\mathbf{Q}_{Y'_I} \to \mathbf{Q}_{Y'_J}$ for $I \subset J$ with $|I| = |J| - 1$, see also Remark (3.5)(iv) below.)

In case $k = n-1$, we may assume $r = 2$ replacing Y with an affine open subvariety if necessary. In this case, X is equisingular along $Y'_{\{1,2\}}$ and has an ordinary double point by restricting to a transversal slice to $Y'_{\{1,2\}}$. So it is enough to show that the restriction of $\mathcal{H}^j\mathrm{IC}_X$ to $Y'_{\{1,2\}}$ is a constant sheaf for any j. Then the assertion is proved by taking the blow-up of X along $Y'_{\{1,2\}}$ which gives a resolution of singularities and using the direct image of the constant sheaf since the intersection complex is a direct factor of the direct image by the decomposition theorem [3].

Lemma 3.3. *Let Z be a complex manifold, and g be a holomorphic function on Z such that $g^{-1}(0)$ is a reduced divisor with simple normal crossings. Let $X' = Z \times \mathbf{C}^2$, and $h = g + z_1 z_2$ on X' where z_1, z_2 are the coordinates of $\mathbf{C}^2$. Then we have a canonical isomorphism compatible with the action of N*

$$\varphi_{h,1}(\mathbf{Q}_{X'}[\dim X']) = \varphi_{g,1}(\mathbf{Q}_Z[\dim Z])(-1).$$

Proof. This is a special case of the Thom-Sebastiani theorem for Hodge modules which was shown in an unpublished manuscript of the second author which was typed at RIMS in 1990. For the proof of Theorem 5 it is enough to show it for the underlying perverse sheaves with $\mathbf{C}$-coefficients since the N-primitive part of each graded piece is a direct sum of the constant sheaves on the strata in this case. Then the assertion also follows from [24], 4.1.

3.4. Proof of Corollary 5. By Theorem 5 the E_1-term $E_1^{-i,j+i}$ is the direct sum of

$$H^{j-i+n-1-2l}(Y_I)(-i-l) \quad \text{and} \quad H^{j-i+n-1-2l-2}(Y'_{I'})(-i-l-1)$$

over I, I' satisfying respectively the conditions

$$|I| = i+2l+1, \quad |I'| = i+2l+2,$$

where l satisfies the conditions $i+l \geq 0$, $i \geq 0$, and

$$\begin{array}{lll} j \leq n-1-i-2l, & j \geq -n+1+i+2l & \text{for } Y_I, \\ j \leq n-1-i-2l-2, & j \geq -n+1+i+2l+2 & \text{for } Y'_{I'}. \end{array} \tag{3.4.1}$$

Note that $\dim Y_I = n-1-i-2l$ and $\dim Y_{I'} = n-1-i-2l-2$.

Let ${}'H^\bullet(Y_I)$ denote the orthogonal complement of the middle primitive cohomology in $H^\bullet(Y_I)$ if $\dim Y_I \neq 0$, and the image of the canonical morphism $\mathbf{Q} \to H^0(Y_I)$ if $\dim Y_I = 0$. Note that ${}'H^j(Y_I) = H^j(Y_I)$ if $j \neq \dim Y_I$. Define ${}'H^\bullet(Y'_I)$ similarly. The differential $d_1^{p,q}$ is expressed by using the restriction and Gysin morphisms up to constant multiples. (This is shown by using the extension classes at the level of sheaves, see Remark (3.5)(iv) below and also [20].) Then ${}'H^\bullet(Y_I)$ and ${}'H^\bullet(Y'_I)$ define a subcomplex ${}'E_1^{\bullet,\bullet}$ of $E_1^{\bullet,\bullet}$. So it is enough to show that its cohomology ${}'E_2^{-i,j+i}$ satisfies

$${}'E_2^{-i,i} = \bigoplus^m \mathbf{Q}((1-n-i)/2) \quad \text{if } i+n-1 \text{ is even and } |i| \leq n-1, \tag{3.4.2}$$

where $m = \binom{r-1}{n}$. Note that ${}'E_2^{-i,i+j} = 0$ for $i+j+n-1$ odd. (Using (3.4.2), the assertion can be reduced to the case $d_k = 1$ for any $k > 0$. Indeed, ${}'E_1^{\bullet,\bullet}$ is independent of the d_k as long as r is fixed, where the differential can be neglected by using the Euler characteristic as below.)

Take $k \in \mathbf{Z}$, and consider the subcomplex ${}'E_1^{\bullet-k,k}$. This is defined by the direct factors of ${}'E_1^{-i,i+j}$ with

$$i+j = k.$$

Here we may assume $k-n-1$ is even since the complex vanishes otherwise. It is enough to calculate the Euler characteristic of ${}'E_1^{\bullet-k,k}$ since we have for $j \neq n-1$

$$H^j(X_\infty, \mathbf{Q}) = \mathbf{Q}(-j/2) \text{ if } j \text{ is even, and } 0 \text{ otherwise.}$$

By the self-duality of $(\psi_{f,1}, N)$, the E_1-complex is self-dual, and we may assume $k \geq 0$.

Let I^l denote the image filtration on ${}'\!E_1$ defined by $\mathrm{Im}\, N^l$. The action of N on ${}'\!E_1$ is defined so that the index l increases by 1 and ${}'\!E_1/I^1$ is isomorphic to the primitive part. Thus the index l is constant on the graded complex $\mathrm{Gr}_I^l\, {}'\!E_1^{\bullet-k,k}$. We will use the index i instead of j so that the complex will be denoted by $\mathrm{Gr}_I^l\, {}'\!E_1^{-\bullet,k}$ (because of the relation $|I| = i + 2l + 1$, etc.) Note that the differential decreases the index i by 1.

Since it is sufficient to calculate the Euler characteristic of $\mathrm{Gr}_I^l\, {}'\!E_1^{-\bullet,k}$, we may modify the differential as we like. So we may calculate it separately for Y_I and Y'_I. We first consider the complex consisting only of the cohomology of Y_I, which will be denoted by $(\mathrm{Gr}_I^l\, {}'\!E_1^{-\bullet,k})_Y$. Let $K_\bullet^{(r)}$ denote the Koszul complex associated to r morphisms which are the identity on $\mathbf{Q}$, where $\dim K_i^{(r)} = \binom{r}{i}$. Since $|I| = i+2l+1$ corresponds to the index i of the Koszul complex up to a shift, we may assume (by modifying the differential of $(\mathrm{Gr}_I^l\, {}'\!E_1^{-\bullet,k})_Y$ if necessary)

$$(\mathrm{Gr}_I^l\, {}'\!E_1^{-\bullet,k})_Y \cong (\sigma_{\leq q}\sigma_{\geq p} K_\bullet^{(r)})[-2l-1],$$

where $\sigma_{\geq p}$ is the truncation which preserves the the components of degree $\geq p$ and replace it with 0 for the degree $< p$ (and similarly for $\sigma_{\leq q}$), see [5]. Note that for $p < q$

$$\dim H_i(\sigma_{\leq q}\sigma_{\geq p} K_\bullet^{(r)}) = \begin{cases} \binom{r-1}{p-1} & \text{if } i = p, \\ \binom{r-1}{q} & \text{if } i = q\,, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4.3}$$

(This follows from the well-known formula $\binom{r}{i} = \binom{r-1}{i} + \binom{r-1}{i-1}$ together with the acyclicity of $K_\bullet^{(r)}$ by decomposing it into the short exact sequences using $\mathrm{Ker}\, d = \mathrm{Im}\, d$.)

The numbers p, q are determined by using the conditions in (3.4.1) for Y_I, and we get

$$p = l + 1, \quad q = l + (n + 1 + k)/2.$$

Moreover the range of l is given by

$$0 \leq l \leq A := (n - 1 - k)/2,$$

which implies $k \leq n - 1$. Then we get the ranges of p, q (when l varies)

$$1 \leq p \leq A + 1, \quad A + k + 1 \leq q \leq n.$$

We apply a similar argument to $(\mathrm{Gr}_I^l\, {}'\!E_1^{-\bullet,k})_{Y'}$ which consists of the cohomology of Y'_I. Since $|I'| = i + 2l + 2$, we have

$$(\mathrm{Gr}_I^l\, {}'\!E_1^{-\bullet,k})_{Y'} \cong (\sigma_{\leq q}\sigma_{\geq p} K_\bullet^{(r)})[-2l-2],$$

where the shift of the Koszul complex is different from the above complex by 1. Applying further a similar argument we get

$$p = l + 2, \quad q = l + (n+1+k)/2, \quad 0 \le l \le A-1,$$
$$2 \le p \le A+1, \quad A+k+1 \le q \le n-1, \quad k \le n-3.$$

If $k \le n-3$, we take the Euler characteristic of ${}'\!E_1^{-\bullet,k}$, and get cancelations for

$$2 \le p \le A+1 \quad \text{and} \quad A+k+1 \le q \le n-1.$$

Thus, using (3.4.3), only $\binom{r-1}{0}$ for $p = 1$ and $\binom{r-1}{n}$ for $q = n$ remain . The first term corresponds to $H^{n-1+k}(X_\infty)$ if $k > 0$, and to the orthogonal complement of the middle primitive part if $k = 0$. So we get the desired assertion. If $k = n-1$, then the range of l consists only of $\{0\}$ for $(\mathrm{Gr}_I^l\,{}'\!E_1^{-\bullet,k})_Y$, and is empty for $(\mathrm{Gr}_I^l\,{}'\!E_1^{-\bullet,k})_{Y'}$. So we get the same conclusion.

Remarks 3.5. (i) For the proof of Corollary 5 we may assume $d_k = 1$ for any $k > 0$, see a remark after (3.4.2). Set $d := d_0 = r$. Let $e_n(d)$ denote the Euler number of a smooth hypersurface Z of degree d in $\mathbf{P}^n$ for $n, d \ge 2$ (e.g. $Z = Y_0$ or Y_I'). It is well-known that

$$e_n(d) = -\textstyle\sum_{i=0}^{n-1} \binom{n+1}{i}(-d)^{n-i} = n+1+((1-d)^{n+1}-1)/d.$$

This can be verified by using an inductive formula $e_n(d) = nd - (d-1)e_{n-1}(d)$ which is easily shown for Fermat hypersurfaces. For $n \ge 1$, set

$$P_n(d) := \dim \widetilde{H}^{n-1}_{\mathrm{prim}}(Z) = (d-1)((d-1)^n - (-1)^n)/d,$$

where $H^{n-1}_{\mathrm{prim}}(Z)$ denotes the primitive cohomology, and the last equality for $n \ge 2$ follows from the above formula. By the argument using ${}'\!E_1^{\bullet,\bullet}$ in (3.4), the direct sums in Corollary 5 are direct factors of the N-primitive part. So Corollary 5 is equivalent to the numerical equality (0.4). In the case $d_k = 1$ $(k > 0)$ and $d_0 = d$, the latter becomes

$$P_n(d) = \textstyle\sum_{0\le k\le n-3}(k+1)\binom{d}{k+2}P_{n-k-2}(d) + n\binom{d-1}{n}.$$

It is possible to prove this by a direct calculation if n is quite small (e.g. if $n \le 5$).

(ii) Let $C(n+1, d, j)$ denote the integers such that

$$(t + \cdots + t^{d-1})^{n+1} = \textstyle\sum_{j=n+1}^{(n+1)(d-1)} C(n+1,d,j)t^j.$$

Up to the multiplication by t^{n+1}, this coincides with the Poincare polynomial of the graded module

$$\mathbf{C}[x_0,\dots,x_n]/(x_0^{d-1},\dots,x_n^{d-1}).$$

Using a Koszul complex which is associated with the multiplication by x_i^{d-1} and gives a free resolution of the above module, we get as is well-known to the specialists

$$C(n+1,d,j) = \textstyle\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}\binom{j-1-k(d-1)}{n}.$$

Here $\binom{p}{q} = 0$ if $q < 0$ or $p - q < 0$.

On the other hand, we have by a well-known result of Griffiths [12]

$$C(n+1,d,di) = \dim \mathrm{Gr}_F^{n-i} H^{n-1}_{\mathrm{prim}}(Z,\mathbf{C}) \quad \text{for } i \in [1,n].$$

Using these, we can calculate the Hodge numbers of the limit mixed Hodge structure in Corollary 5.

(iii) With the notation of Theorem 5 we have the weight spectral sequence for the vanishing cycles

$$E_1^{-k,j+k} = H^j(X_0, \mathrm{Gr}^W_{k+n-1}\varphi_f(\mathbf{Q}_X[n])) \Rightarrow H^j(X_0, \varphi_f(\mathbf{Q}_X[n])),$$

degenerating at E_2, and $E_1^{-i,j+i}$ is given by the direct sum of

$$\bigoplus_{l\ge\max(-i,1)}\Big(\bigoplus_{|I|=i+2l+1} H^{j+n-|I|}(Y_I)(-i-l)\Big)$$

and

$$\bigoplus_{l\ge\max(-i,0)}\Big(\bigoplus_{|I|=i+2l+2} H^{j+n-|I|-1}(Y'_I)(-i-l-1)\Big).$$

(iv) Let $i : X \hookrightarrow Y$ be a closed immersion of smooth varieties. Assume $\mathrm{codim}_Y X = 1$. Set $n = \dim Y$. Since $i^*\mathbf{Q}_Y = \mathbf{Q}_X$ and $i^!\mathbf{Q}_Y = \mathbf{Q}_X(-1)[-2]$, the adjunction for i implies

$$\mathrm{Ext}^1(\mathbf{Q}_Y[n], \mathbf{Q}_X[n-1]) = \mathrm{Ext}^1(\mathbf{Q}_X[n-1], \mathbf{Q}_Y(1)[n]) = \mathrm{Hom}(\mathbf{Q}_X, \mathbf{Q}_X).$$

(v) For a quasi-semistable degeneration, it would be possible to get a semistable model by repeating the blow-ups along the center $\{t = y_k = 0\}$ in the notation of (3.1.1). Indeed, the strict transform of $\{y_1 \cdots y_k = y_n t\}$ becomes $\{y_1 \cdots y_{k-1} = y_n t\}$ or $\{y_1 \cdots y_k = y_n\}$ by replacing t with ty_k or y_k with ty_k, see [13] for the case $r = 2$. If $r = 2$ we can prove Theorem (3.2) using this blow-up, see [20].

4. Application

In this section we prove Theorem 6 which improves [15], Cor. 6.4.

4.1. Proof of Theorem 6. By assumption $L = L_1 \otimes L_2$ with L_1, L_2 very ample. Moreover L_2 is a tensor product of two very ample line bundles if $Y = \mathbf{P}^n$ with n even. Let $Y_i \in |L_i|$ $(i = 0, 1, 2)$ be general smooth members where $L_0 := L$. We may assume that their union is a divisor with normal crossings. Let $g_i \in \Gamma(X, L_i)$ defining Y_i for $i = 0, 1, 2$. Applying the construction of Theorem 5, we get a smoothing $f : X \to \mathbf{C}$ of $Y_1 \cup Y_2$ such that

$$X_t := f^{-1}(t) = \{g_1 g_2 = t g_0\} \subset X.$$

In particular, $X_0 = Y_1 \cup Y_2$ and $X_1 = Y_0$. The singular locus Σ of X is contained in $X_0 = \{t = 0\}$ and coincides with the intersection of Y_0, Y_1, Y_2, see (3.1.2). Set $Z = Y_1 \cap Y_2$. Note that $\dim Y_i = n - 1$, $\dim Z = n - 2$ and $\dim \Sigma = n - 3$. By Theorem 5 we have the weight spectral sequence

$$E_1^{-k,j+k} = H^j(X_0, \mathrm{Gr}^W_{k+n-1}\psi_{f,1}\mathrm{IC}_X\mathbf{Q}) \Rightarrow H^{j+n-1}(X_\infty, \mathbf{Q}),$$

degenerating at E_2, and its E_1-terms are given as follows:

$$E_1^{-k,j+k} = \begin{cases} H^{j+n-2}(Z)(-1) & \text{if } k=1, \\ \bigoplus_{i=1,2} H^{j+n-1}(Y_i) \oplus H^{j+n-3}(\Sigma)(-1) & \text{if } k=0, \\ H^{j+n-2}(Z) & \text{if } k=-1, \\ 0 & \text{otherwise.} \end{cases}$$

(It is also possible to get this spectral sequence by the same argument as in [20], 4.4 using the blow-up along $Y_0 \cap Y_2$ studied in [13].) The differential $d_1^{p,q}$ is expressed by the restriction and Gysin morphisms associated to the inclusions $\Sigma \to Z$ and $Z \to Y_i$ up to constant multiples, see see Remark (3.5)(iv) below. Set

$$\alpha_r^{j,k} = \dim E_r^{-k,j+k},$$
$$\gamma^j(V) = \dim H^{j+\dim V}(V) \ \text{ for } V = Y, Y_i, Z, \Sigma.$$

There are symmetries

$$\alpha_1^{j,k} = \alpha_1^{-j,k} = \alpha_1^{j,-k} = \alpha_1^{-j,-k}, \quad \gamma^{-j}(V) = \gamma^j(V).$$

We have to show

$$\gamma^0(Y_0) = \textstyle\sum_{|k|\le 1} \alpha_2^{0,k} > \gamma^{-1}(Y), \tag{4.1.1}$$

where the first equality follows from the E_2-degeneration of the above spectral sequence. We will show (4.1.1) by induction on n.

The weak Lefschetz theorem implies

$$\begin{aligned} \gamma^0(\Sigma) \ge \gamma^{-1}(Z) = \gamma^{-2}(Y_i) = \gamma^{-3}(Y), \\ \gamma^0(Z) \ge \gamma^{-1}(Y_i) = \gamma^{-2}(Y), \\ \gamma^0(Y_i) \ge \gamma^{-1}(Y). \end{aligned} \tag{4.1.2}$$

From the above spectral sequence we get then

$$\begin{aligned} &\alpha_2^{0,0} - \gamma^{-1}(Y) \ge \alpha_1^{0,0} - \alpha_1^{-1,1} - \alpha_1^{1,-1} - \gamma^{-1}(Y) \\ &= \gamma^0(Y_1) - \gamma^{-1}(Y) + \gamma^0(Y_2) - \gamma^{-3}(Y) + \gamma^0(\Sigma) - \gamma^{-3}(Y), \end{aligned} \tag{4.1.3}$$

using $\alpha_1^{-1,1} = \alpha_1^{1,-1} = \gamma^{-1}(Z) = \gamma^{-3}(Y)$. We have moreover

$$\alpha_2^{0,-1} \ge \gamma^0(Z) - \gamma^{-2}(Y) = \gamma^0(Z) - \gamma^{-1}(Y_1). \tag{4.1.4}$$

Indeed, by the weak Lefschetz theorem, the image of $H^{n-2}(Y_i)$ $(i=1,2)$ in $H^{n-2}(Z)$ under the differential d_1 of the spectral sequence is contained in the image of the canonical injection $H^{n-2}(Y) \hookrightarrow H^{n-2}(Z)$ by the restriction morphism. Moreover, the image of $H^{n-4}(\Sigma)(-1)$ in $H^{n-2}(Z)$ is also contained in the image of the above canonical injection since it is contained in the image of the action of $c_1(L_0)$ using the bijectivity of the restriction morphism $H^{n-4}(Z) \to H^{n-4}(\Sigma)$ in the case $n \ge 4$. (Note that the action of $c_1(L_0)$ is compatible with the restriction morphism.) So (4.1.4) follows.

Thus, in order to prove (4.1.1) using (4.1.2–4), it is enough to show either

$$\gamma^0(\Sigma) > \gamma^{-1}(Z) = \gamma^{-3}(Y) \quad \text{or} \quad \gamma^0(Z) - \gamma^{-1}(Y_1) > 0. \tag{4.1.5}$$

Using the first condition of (4.1.5), the assertion is reduced to the case where n, Y and Y_0 are respectively replaced by $n-2$, Z and Σ. So we get the assertion in the n odd case by induction on n. Indeed, in case $n=3$ we have $\#|\Sigma|\geq 2$ since it coincides with the intersection number $Y_0\cdot Y_1\cdot Y_2$ where Y_0 can be replaced by Y_1+Y_2 as algebraic cycles and Y_1, Y_2 are very ample by assumption.

We may now assume n even. In the case where $Y=\mathbf{P}^n$ with n even so that L is a tensor product of three very ample line bundles, the assertion then follows from the n odd case using the last condition of (4.1.5) and replacing Y with Y_1. In the other case, the assertion is reduced finally to the case $n=2$ by induction on n using the first condition of (4.1.5) repeatedly where n, Y and Y_0 are respectively replaced by $n-2$, Z and Σ in each inductive step. We have then $\Sigma=\emptyset$, $\dim Z=0$ and Y_i is a connected curve where the image of $H^0(Y_i)\to H^0(Z)$ is the diagonal for $i=1,2$ and its cokernel is nonzero if $\#|Z|>1$. The last condition is satisfied after replacing Y with Z repeatedly if the intersection number $Y_1^{n/2}\cdot Y_2^{n/2}$ is bigger than 1 for the original $Y_1, Y_2\subset Y$. So the assertion is reduced to the following lemma (which would be known to specialists at least in the case $D_i=D_j$ for any i,j).

Lemma 4.2. *Let Y be a smooth complex projective variety of dimension $n\geq 2$, and D_i be very ample divisors for $i=1,\dots,n$. Assume the intersection number $D_1\cdots D_n$ is 1. Then $Y=\mathbf{P}^n$ and the D_i are hyperplanes.*

Proof. Note first that the assertion is easy if $D_i\in|L|$ for some very ample line bundle L independent of i. Indeed, take a linear subsystem generated by general $n+1$ hyperplane sections and defining a morphism $Y\to\mathbf{P}^n$. Then the assumption implies that its fiber is one point over any $s\in\mathbf{P}^n$ (taking n hyperplanes in $\mathbf{P}^n$ whose intersection is s). So the assertion follows.

We prove the general case by induction on $n\geq 1$. If $n=1$, the intersection number is interpreted as the degree of a zero-cycle. Then the assertion is well known. Assume $n\geq 2$. We first show that the D_i are isomorphic to $\mathbf{P}^{n-1}$. For this it is enough to show that the D_i are smooth using the inductive hypothesis. But it is well known (and is easy to show) that the intersection number cannot be 1 if some D_i has a singular point. Note that D_i may be replaced by any member of the linear system $|\mathcal{O}_Y(D_i)|$ and the D_i are very ample.

We now consider a Lefschetz pencil $f: X\to\mathbf{P}^1$ associated to the very ample line bundle $\mathcal{O}_Y(D_n)$. Its fibers are *all* isomorphic to $\mathbf{P}^{n-1}$ by the above argument. So we get

$$h^{2i}(X)=2 \text{ if } i=1,\dots,n-1, \text{ and } h^j(X)=0 \text{ if } j \text{ is odd.}$$

Since the morphism $\rho: X\to Y$ is the blow-up along a smooth center C which has codimension 2, we have

$$h^{2i}(X)=h^{2i}(Y)+h^{2i-2}(C)\quad\text{for any } i,$$

and $h^{2i}(Y)$ and $h^{2i-2}(C)$ are nonzero for $i=1,\dots,n-1$ since Y, C are projective. We get thus

$$h^{2i}(Y)=1 \text{ if } i=1,\dots,n-1, \text{ and } h^j(Y)=0 \text{ if } j \text{ is odd.}$$

Moreover, $H^2(Y,\mathbf{Z})$ is torsion-free since so is $H^2(X,\mathbf{Z})$ and there is a long exact sequence

$$0 \to H^2(Y,\mathbf{Z}) \to H^2(X,\mathbf{Z}) \to H^0(C,\mathbf{Z})(-1) \to .$$

(The last sequence is induced by the truncation τ on the direct image $\mathbf{R}\rho_*\mathbf{Z}_X$ since $R^i\rho_*\mathbf{Z}_X$ is $\mathbf{Z}_Y$ if $i=0$, $\mathbf{Z}_C(-1)$ if $i=2$, and 0 otherwise.) So we get $\mathrm{Pic}(Y)=\mathbf{Z}[D]$ where D is an ample divisor, and $[D_i]=m_i[D]$ for positive integers m_i. Then the assumption implies $m_i=1$ for any i, and the assertion is reduced to the first case.

Remarks 4.3. (i) The equivalence between the non-vanishing and the non-surjectivity in Theorem 6 follows from the Picard-Lefschetz formula [17] and the global invariant cycle theorem [5], see [15] (and [16] for a complex analytic argument).

(ii) If $Y=\mathbf{P}^n$ with n even and $L=\mathcal{O}_{\mathbf{P}^n}(2)$, then $H^{n-1}(Y_s)=0$ for a smooth member Y_s of $|L|$ and we have the surjectivity of i_s^*. So Theorem 6 is optimal (i.e. we cannot take $k=2$) in the case Y is projective space of even dimension. Among the other cases where Theorem 6 holds with $k=2$, there are some cases where we cannot take $k=1$. For example, if Y is a projective hypersurface of degree $d\le 2$ and $L=\mathcal{O}_Y(1)$, then we have the surjectivity of i_s^* except for the case where $\dim Y$ is odd and $d=2$. Recently N. Fakhruddin informed us that there are less trivial examples. For instance, if Y is a ruled surface over $\mathbf{P}^1$, then we have a very ample divisor which is a sum of a section C and the pull-back of a very ample divisor on $\mathbf{P}^1$. In this case any member of the linear system $|L|$ is a section (since its intersection number with any fiber is 1) and we have the surjectivity of i_s^* for $k=1$.

(iii) The non-surjectivity of the restriction morphism i_s^* in Theorem 6 is equivalent to the condition that $H^n(Y\setminus Y_s)$ is not pure of weight n by Poincaré duality and the weight spectral sequence [5]. This implies examples of smooth affine varieties $Y\setminus Y_s$ whose cohomology is not pure and which are formal in the sense of [8], see [19], Cor. 10.5, a).

(iv) Let $\pi:\mathcal{Y}\to S:=(\mathbf{P}^N)^\vee$ denote the universal family of hyperplane sections as in a remark after Theorem 6. By [3] we have the decomposition

$$\mathbf{R}\pi_*(\mathbf{Q}_{\mathcal{Y}}[\dim\mathcal{Y}])\cong\textstyle\bigoplus_{i,j}E_j^i[-i],$$

where E_j^i denotes the direct sum of the direct factors of ${}^pR^i\pi_*(\mathbf{Q}_{\mathcal{Y}}[\dim\mathcal{Y}])$ having strict support of codimension j. In [4] it is proved that

$$E_j^i=0 \quad \text{for } \ ij\ne 0.$$

Moreover, N. Fakhruddin proved in Appendix of loc. cit.

$$E_j^0=0 \ \text{ for } j\ge 1 \text{ if } L=L'^{\otimes d} \text{ with } L' \text{ ample and } d\gg 1.$$

Here it is sufficient to assume $d\ge 2n-1$ if L' is very ample. Indeed, let $\mathcal{I}_Z\subset\mathcal{O}_Y$ be the reduced ideal sheaf of $Z=\{z_1,\dots,z_m\}\subset Y$. By the same argument as in loc. cit., the assertion can be reduced to the surjection

$$H^0(Y,L'^{\otimes d})\twoheadrightarrow H^0(Y,L'^{\otimes d}\otimes_{\mathcal{O}_Y}(\mathcal{O}_Y/\mathcal{I}_Z^2)) \quad \text{for } \ d\ge 2m-1.$$

For the last surjection, take $H_{i,j} \in |L|$ for $i \in [1,m]$, $j \in [0,n]$ such that $z_i \notin H_{i,0}$ and the $H_{i,j}$ for $j \in [1,n]$ contain z_i and form a divisor with normal crossings at z_i. Take further $H'_i \in |L^{m-1}|$ such that $z_i \notin H'_i$ and $z_j \in H'_i$ for $j \neq i$. Then the surjection follows by using $H_{i,j} + 2H'_i \in |L^{2m-1}|$ for $i \in [1,m]$, $j \in [0,n]$. Note that Theorem 6 implies (assuming $Y \neq \mathbf{P}^n$)

$$E_1^0 = 0 \text{ if } L = L'^{\otimes d} \text{ with } L' \text{ very ample and } d \geq 2.$$

(v) Let Y_s be a smooth fiber of π in Remark (iv). Set $n = \dim Y$. Using the Picard-Lefschetz formula, we have the orthogonal decomposition

$$H^{n-1}(Y_s) = H^{n-1}(Y_s)^{\text{inv}} \oplus H^{n-1}(Y_s)^{\text{van}},$$

where $H^{n-1}(Y_s)^{\text{inv}} = \text{Im}(H^{n-1}(Y) \to H^{n-1}(Y_s))$ and $H^{n-1}(Y_s)^{\text{van}}$ is generated by the vanishing cycles (this is closely related to the hard Lefschetz theorem, see [7], [16].)

We have

$$H^{n-1}(Y_s)^{\text{van}} \subset H^{n-1}_{\text{prim}}(Y_s),$$

where $H^{n-1}_{\text{prim}}(Y_s)$ denotes the primitive part. Indeed, the non-primitive part is contained in the invariant part and the assertion follows by taking their orthogonal complements.

Let $f : X \to \mathbf{P}^1$ be a Lefschetz pencil where $\rho : X \to Y$ is the blow-up along a smooth center Z which is the intersection of two general hyperplane sections. We have the perverse Leray spectral sequence

$$E_2^{i.j} = H^i(\mathbf{P}^1, {}^pR^j f_*(\mathbf{Q}_X[n])) \Rightarrow H^{i+j+n}(X),$$

degenerating at E_2 by the decomposition theorem [3]. Note that

$$\begin{aligned} E_2^{-1,j+1} &= \text{Im}(\tilde{i}_s^* : H^{j+n}(X) \to H^{j+n}(Y_s)), \\ E_2^{1,j-1} &= \text{Im}((\tilde{i}_s)_* : H^{j+n-2}(Y_s)(-1) \to H^{j+n}(X)), \end{aligned}$$

where $\tilde{i}_s : Y_s \hookrightarrow X$ is the inclusion. Moreover we have the decomposition

$$H^{j+n}(X) = H^{j+n}(Y) \oplus H^{j+n-2}(Z)(-1),$$

so that $\tilde{i}_s^*, (\tilde{i}_s)_*$ are expressed by using $i_s^*, (i_s)_*, i_s'^*, (i'_s)_*$ where $i_s : Y_s \hookrightarrow Y$, $i'_s : Z \hookrightarrow Y_s$ are the inclusions, see [15]. Using the weak Lefschetz theorem, we get then a canonical isomorphism as Hodge structures

$$E_2^{0,0} = H^n_{\text{prim}}(Y) \oplus H^{n-2}(Z)^{\text{van}}(-1).$$

Let $D \subset S$ denote the discriminant of π in Remark (iv). In the case $\text{codim}_S D > 1$, ${}^pR^0 f_*(\mathbf{Q}_X[n])$ is constant so that $E_2^{0,0} = 0$, and we get by using the above formula

$$H^n_{\text{prim}}(Y) = H^{n-1}(Y_s)^{\text{van}} = H^{n-2}(Z)^{\text{van}} = 0.$$

Note that its converse is also true. Indeed, the equivalence can be shown by using the following (see [16], 3.5.3)

$$\chi(Y) - 2\chi(Y_s) + \chi(Z) = (-1)^n r,$$

where r is the number of the critical values of the Lefschetz pencil, i.e. the intersection number of D with a general line. The weak Lefschetz theorem implies that the left-hand side coincides up to a sign with a sum of three non-negative numbers

$$\big(b_n(Y)-b_{n-2}(Y)\big)+2\big(b_{n-1}(Y_s)-b_{n-1}(Y)\big)+\big(b_{n-2}(Z)-b_{n-2}(Y_s)\big).$$

References

[1] Barlet, D. and Maire, H.-M., Poles of the current $|f|^{2\lambda}$ over an isolated singularity, Internat. J. Math. 11 (2000), 609–635.

[2] Barlet, D. and Maire, H.-M., Non-trivial simple poles at negative integers and mass concentration at singularity, Math. Ann. 323 (2002), 547–587.

[3] Beilinson, A.A., Bernstein, J. and Deligne, P., Faisceaux pervers, Astérisque 100, Soc. Math. France, Paris, 1982.

[4] Brosnan, P., Fang, H., Nie, Z. and Pearlstein, G.J., Singularities of admissible normal functions (with an appendix by Najmuddin Fakhruddin), Invent. Math. 177 (2009), 599–629.

[5] Deligne, P., Théorie de Hodge II, Publ. Math. IHES, 40 (1971), 5–58.

[6] Deligne, P., Le formalisme des cycles évanescents, in SGA7 XIII and XIV, Lect. Notes in Math. 340, Springer, Berlin, 1973, pp. 82–115 and 116–164.

[7] Deligne, P., Conjecture de Weil II, Publ. Math. IHES, 52 (1980), 137–252.

[8] Deligne, P., Griffiths, P., Morgan, J. and Sullivan, D., Real homotopy theory of Kähler manifolds, Inv. Math. 29 (1975), 245–274.

[9] Durfee, A.H. and Saito, M., Mixed Hodge structures on the intersection cohomology of links, Compos. Math. 76 (1990), 49–67.

[10] Goresky, M. and MacPherson, R., Intersection homology theory, Topology 19 (1980), 135–162.

[11] Green, M., Griffiths, P. and Kerr, M., Néron models and limits of Abel-Jacobi mappings (preprint).

[12] Griffiths, P., On the period of certain rational integrals I, II, Ann. Math. 90 (1969), 460–541.

[13] Griffiths, P. and Harris, J., On the Noether-Lefschetz theorem and some remarks on codimension two cycles, Math. Ann. 271 (1985) 31–51.

[14] Hamm, H., Lokale topologische Eigenschaften komplexer Räume, Math. Ann. 191 (1971), 235–252.

[15] Katz, N., Etude cohomologique des pinceaux de Lefschetz, in Lect. Notes in Math., vol. 340, Springer Berlin, 1973, pp. 254–327.

[16] Lamotke, K., The topology of complex projective varieties after S. Lefschetz, Topology 20 (1981), 15–51.

[17] Lefschetz, S., L'analysis situs et la géométrie algébrique, Gauthier-Villars, Paris, 1924.

[18] Massey, D.B., Intersection cohomology, monodromy, and the Milnor fiber, preprint (arXiv: math/0404312).

[19] Morgan, J.W., The algebraic topology of smooth algebraic varieties, Inst. Hautes Etudes Sci. Publ. Math. No. 48 (1978), 137–204.

[20] Otwinowska, A. and Saito, M., Monodromy of a family of hypersurfaces containing a given subvariety, Ann. Sci. Ecole Norm. Sup. (4) 38 (2005), 365–386.

[21] Saito, M., Hodge structure via filtered D-modules, Astérisque 130 (1985), 342–351.

[22] Saito, M., Modules de Hodge polarisables, Publ. RIMS, Kyoto Univ. 24 (1988), 849–995.

[23] Saito, M., Mixed Hodge modules, Publ. RIMS, Kyoto Univ. 26 (1990), 221–333.

[24] Saito, M., On microlocal b-function, Bull. Soc. Math. France 122 (1994), 163–184.

[25] Schmid, W., Variation of Hodge structure: The singularities of the period mapping, Inv. Math. 22 (1973), 211–319.

[26] Steenbrink, J.H.M., Limits of Hodge structures, Inv. Math. 31 (1975/76), 229–257.

[27] Steenbrink, J.H.M., Monodromy and weight filtration for smoothings of isolated singularities, Compos. Math. 97 (1995), 285–293.

[28] Torrelli, T., Intersection homology D-Module and Bernstein polynomials associated with a complete intersection, preprint (arXiv:0709.1578) to appear in Publ. RIMS.

Laboratoire J.A. Dieudonné, UMR du CNRS 6621, Université de Nice-Sophia Antipolis, Parc Valrose, 06108 Nice Cedex 02, France
E-mail address: Alexandru.DIMCA@unice.fr

RIMS Kyoto University, Kyoto 606-8502 Japan
E-mail address: msaito@kurims.kyoto-u.ac.jp